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On the zero-one law for connectivity in one-dimensional geometric random graphs

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On zero-one laws for connectivity in one-dimensional geometric random graphs

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***Abstract*— We consider the geometric random graph where n points are distributed uniformly and independently on the unit interval $[0, 1]$. Using the method of first and second moments, we provide a simple proof of the "zero-one" law for the property of graph connectivity under the asymptotic regime created by having n become large and the transmission range scaled appropriately with n.**

I. Introduction

INTEREST in geometric random graphs as models for wireless networks has been stimulated to a great extent by the paper of Gupta and Kumar [10]. Here, we consider a one-dimensional random graph model which has been studied recently by a number of authors, e.g., see [3], [5], [6], [7], [8], [9], [14]: The network comprises n points (or nodes) which are distributed uniformly and independently on the unit interval $[0, 1]$. Two nodes are said to communicate with each other if their distance is less than some given transmission range $\tau > 0$. Let $P(n; \tau)$ denote the probability that the network (as induced graph) is connected.

Randomizing node locations makes it possible for many properties (including graph connectivity) to display a typical behavior when n becomes large and the transmission range τ is scaled appropriately, i.e., is made to depend on n through scalings or range functions $\tau : \mathbb{N}_0 \to \mathbb{R}_+ : n \to \tau_n$. Typical behavior reveals itself through "zero-one" laws whereby a given graph property occurs (resp. does not occur) with a very high probability (as n becomes large) depending on how the scaling used deviates from a *critical* scaling $\tau^\star$ (which is property dependent).

For the property of graph connectivity, the critical scaling is known to be $\tau_n^\star = \frac{\log n}{n}$, e.g., see [1], [14], with the following rough meaning: For n sufficiently large, a communication range τ_n suitably larger (resp. smaller) than $\tau_n^\star$ ensures $P(n; \tau_n) \simeq 1$ (resp. $P(n; \tau_n) \simeq 0$). In these references, the precise technical meaning for suitably larger (resp. smaller) amounts to $\tau_n = c\tau_n^\star$ with $c > 1$ (resp. $0 < c < 1$). In this short note, we strengthen this result by showing that the zero-one law still holds if we allow *much smaller* deviations (than $(c-1)\tau_n^\star$) from the critical scaling $\tau_n^\star$. This is the content of Theorem 2.1 (discussed in the next section) which provides also an early indication of the sharpness of the corresponding phase transition [11]. The proof is self-contained and uses *elementary* arguments based on the method of first and second moments [12, p. 55], an approach widely used in the theory of Erdős-Renyi graphs.

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II. The main result

We start with a sequence $\{X_i, \ i = 1, 2, \ldots\}$ of i.i.d. rvs distributed uniformly in the interval $[0, 1]$. For each $n = 2, 3, \ldots$, we think of $X_1, \ldots, X_n$ as the locations of n nodes (or users), labelled $1, \ldots, n$, in the interval $[0, 1]$. Given a fixed communication range $\tau > 0$, nodes i and j are connected if $|X_i - X_j| \leq \tau$, in which case an undirected edge is said to exist between them.

This notion of edge connectivity gives rise to an undirected geometric random graph, thereafter denoted $\mathbb{G}(n; \tau)$. The graph $\mathbb{G}(n; \tau)$ is said to be *(path) connected* if every pair of users can be linked by at least one path over the edges of the graph, and the probability of graph connectivity is given by

$$P(n; \tau) := \mathbf{P}\left[\mathbb{G}(n; \tau) \text{ is connected}\right]. \qquad (1)$$

While obviouly $P(n; \tau) = 1$ whenever $\tau \geq 1$, we find it convenient to set $P(n; \tau) = 0$ for $\tau < 0$.

The main result of this note is given in Theorem 2.1 below. To prepare for it, we note that there is no loss of generality in writing any range function $\tau : \mathbb{N}_0 \to \mathbb{R}_+$ in the form

$$\tau_n = \frac{1}{n}\left(\log n + \alpha_n\right), \quad n = 1, 2, \ldots. \qquad (2)$$

for some deviation function $\alpha : \mathbb{N}_0 \to \mathbb{R}$.

Theorem 2.1: For any range function $\tau : \mathbb{N}_0 \to \mathbb{R}_+$ in the form (2), it holds that

$$\lim_{n\to\infty} P(n; \tau_n) = \begin{cases} 0 & \textit{if } \lim_{n\to\infty} \alpha_n = -\infty \\ \\ 1 & \textit{if } \lim_{n\to\infty} \alpha_n = +\infty. \end{cases} \qquad (3)$$

Theorem 2.1 identifies the range function $\tau^\star : \mathbb{N}_0 \to \mathbb{R}_+$ given by

$$\tau_n^\star = \frac{\log n}{n}, \quad n = 1, 2, \ldots \qquad (4)$$

as the *critical* scaling defining a threshold or boundary in the space of range functions. However, the conclusion (3) is quite stronger than the one usually discussed in the literature, namely

$$\lim_{n\to\infty} P(n; c\tau_n^\star) = \begin{cases} 0 & \text{if } \ 0 < c < 1 \\ \\ 1 & \text{if } \ 1 < c. \end{cases} \qquad (5)$$

This last result still holds for any range function $\tau : \mathbb{N}_0 \to \mathbb{R}_+$ such that $\tau_n \sim c\tau_n^\star$ for some $c > 0$ – Here and throughout the paper, such asymptotic equivalence is understood with n going to infinity. Either of these equivalent forms is already contained in Theorem 1 by Appel and Russo [1, p. 352]. More recently, Muthukrishnan and Pandurangan [14, Thm.

2.2] obtain (5) by a bin-covering technique. The zero-one law (5) warrants that the threshold function $\tau^\star$ be called a *strong* threshold [13].

To better appreciate the difference between (3) and (5), we write a range function $\tau : \mathbb{N}_0 \to \mathbb{R}_+$ of the form (2) as

$$\tau_n = \tau_n^\star + \frac{\alpha_n}{n}, \quad n = 1, 2, \ldots \tag{6}$$

with corresponding deviation function $\alpha : \mathbb{N}_0 \to \mathbb{R}$. From Theorem 2.1, perturbations $\frac{\alpha_n}{n}$ from the critical threshold yield the one-law (resp. zero-law) provided $\lim_{n\to\infty} \alpha_n = \infty$ (resp. $\lim_{n\to\infty} \alpha_n = -\infty$) with no further constraint on α. Contrast this with (5) where we allow only scalings of the form $\tau_n \sim c\tau_n^\star$ with $c > 0$ and $c \neq 1$, so that $\alpha_n \sim (c-1)\log n$. It is now plain that (5) is indeed implied by (3). Whereas "small" deviations of the form $\alpha_n = \pm \log \log n$ are covered by Theorem 2.1, they are not covered by the zero-one law (5) (since $\alpha_n = o(\log n)$). Consequently, it seems appropriate to call the critical scaling $\tau^\star$ a *very strong* (and not merely a strong) threshold for the property of graph connectivity. This is certainly in line with the very sharp phase transition already aparent from the graphs available in several papers, e.g., see [7], [9], and formally established in [11].

III. Preliminaries

Fix $n = 2, 3, \ldots$ and $\tau > 0$. With the node locations $X_1, \ldots, X_n$, we associate the rvs $X_{n,1}, \ldots, X_{n,n}$ which are the location of these n users arranged in increasing order, i.e., $X_{n,1} \leq \ldots \leq X_{n,n}$ with the convention $X_{n,0} = 0$ and $X_{n,n+1} = 1$. The rvs $X_{n,1}, \ldots, X_{n,n}$ are the *order statistics* associated with the n i.i.d. rvs $X_1, \ldots, X_n$. Also define the spacings

$$L_{n,k} := X_{n,k} - X_{n,k-1}, \quad k = 1, \ldots, n+1.$$

Interest in these quantities derives from the observation that the graph $\mathbb{G}(n;\tau)$ is connected if and only if $L_{n,k} \leq \tau$ for all $k = 2, \ldots, n$, so that

$$P(n;\tau) = \mathbf{P}\left[L_{n,k} \leq \tau,\ k = 2, \ldots, n\right]. \tag{7}$$

It is well known [2, Eq. (6.4.3), p. 135] that for any subset $I \subseteq \{1, \ldots, n\}$, we have

$$\begin{aligned} & \mathbf{P}\left[L_{n,k} > t_k,\ k \in I\right] \\ = & \left(1 - \sum_{k\in I} t_k\right)_+^n, \quad t_k \in [0,1],\ k \in I \end{aligned} \tag{8}$$

with the notation $x_+^n = x^n$ if $x \geq 0$ and $x_+^n = 0$ if $x \leq 0$.

Using (8), it is easy to obtain closed form expression for $P(n;\tau)$. This closed form expression has been rediscovered by several authors, e.g., Godehardt and Jaworski [8, Cor. 1, p. 146], Desai and Manjunath [3] (as Eqn (8) with $z = 1$ and $r = \tau$), Ghasemi and Nader-Esfahani [7] and Gore [9]. See also Devroye's paper [4] for pointers to an older literature.

We conclude this section with some easy convergence facts to be used in the proof of Theorem 2.1: With $0 \leq x < 1$, it is a simple matter to check that

$$\log(1-x) = -\int_0^x \frac{1}{1-t}dt = -x - \psi(x) \tag{9}$$

where we have set

$$\psi(x) := \int_0^x \frac{t}{1-t}dt, \quad 0 \leq x < 1.$$

The mapping $x \to \psi(x)$ is increasing and convex on the interval $[0,1)$ with

$$0 < \psi(x) \leq \frac{x^2}{2(1-x)}, \quad 0 \leq x < 1. \tag{10}$$

Now consider a range function $\tau : \mathbb{N}_0 \to \mathbb{R}_+$ in the form (2). For each $p > 0$, provided $p\tau_n < 1$, the decomposition (9) yields

$$\begin{aligned} (1 - p\tau_n)_+^n &= e^{-n(p\tau_n + \psi(p\tau_n))} \\ &= e^{-p(\log n + \alpha_n)} e^{-n\psi(p\tau_n)} \\ &= n^{-p} e^{-p\alpha_n} e^{-n\psi(p\tau_n)}. \end{aligned} \tag{11}$$

The next two technical lemmas rely on this observation; they will be useful in a number of places.

Lemma 3.1: For any range function $\tau : \mathbb{N}_0 \to \mathbb{R}_+$ *in the form (2) with* $\lim_{n\to\infty} \alpha_n = -\infty$*, we have*

$$\lim_{n\to\infty} \frac{(1-p\tau_n)_+^n}{n^{-p}e^{-p\alpha_n}} = 1, \quad p > 0. \tag{12}$$

Proof. Fix $p > 0$. In view of (11), it is plain that Lemma 3.1 will hold if the assumption $\lim_{n\to\infty} \alpha_n = -\infty$ can be shown to imply

$$p\tau_n < 1 \quad \text{for all sufficiently large } n \tag{13}$$

with

$$\lim_{n\to\infty} n\psi(p\tau_n) = 0. \tag{14}$$

Condition (13) ensures that (11) holds for large enough n.

First, from the assumption $\lim_{n\to\infty} \alpha_n = -\infty$, we note that $\alpha_n < 0$ for large enough n and the form (2) therefore implies $\tau_n \leq \frac{\log n}{n}$ on that range, whence

$$\lim_{n\to\infty} \tau_n = \lim_{n\to\infty} \frac{\alpha_n}{n} = 0$$

since $\lim_{n\to\infty} \frac{\log n}{n} = 0$. This already establishes (13).

Still on that range, the monotonicity of ψ yields

$$n\psi(p\tau_n) \leq n\psi\left(p\frac{\log n}{n}\right)$$

so that

$$n\psi(p\tau_n) \leq \frac{p^2}{2} \cdot \left(1 - p\frac{\log n}{n}\right)^{-1} \cdot \frac{(\log n)^2}{n}$$

by invoking the bound (10). The validity of (14) is now straightforward. ■

Lemma 3.2: Consider a range function $\tau : \mathbb{N}_0 \to \mathbb{R}_+$ *in the form (2). It holds that*

$$\lim_{n\to\infty} n(1-\tau_n)_+^n = \begin{cases} \infty & \text{if } \lim_{n\to\infty}\alpha_n = -\infty \\ 0 & \text{if } \lim_{n\to\infty}\alpha_n = +\infty. \end{cases} \tag{15}$$

Proof. First, we note that

$$n\,(1-\tau_n)_+^n = e^{-\alpha_n} \cdot \frac{(1-\tau_n)_+^n}{n^{-1}e^{-\alpha_n}}, \quad n = 1, 2, \ldots \tag{16}$$

and Lemma 3.1 (with $p = 1$) readily yields the conclusion $\lim_{n\to\infty} n(1-\tau_n)_+^n = \infty$ when $\lim_{n\to\infty} \alpha_n = -\infty$.

We note that $n(1-\tau_n)_+^n = 0$ if $1 \leq \tau_n$, while when $\tau_n \leq 1$, we conclude from (11) that $n(1-\tau_n)_+^n \leq e^{-\alpha_n}$ by the non-negativity of ψ. It is now immediate that $\lim_{n\to\infty} n(1-\tau_n)_+^n = 0$ when $\lim_{n\to\infty} \alpha_n = +\infty$. ∎

IV. A PROOF OF THEOREM 2.1

Fix $n = 2, 3, \ldots$ and τ in $(0, 1)$. For each $i = 1, \ldots, n$, node i is said to be a *breakpoint* node in the random graph $\mathbb{G}(n;\tau)$ whenever (i) it is *not* the leftmost node in $[0, 1]$ and (ii) there is no node in the random interval $[X_i - \tau, X_i]$. The number $C_n(\tau)$ of breakpoint nodes in $\mathbb{G}(n;\tau)$ is given by

$$C_n(\tau) = \sum_{k=2}^{n} \chi_{n,k}(\tau)$$

where the $\{0, 1\}$-valued rvs $\chi_{n,1}(\tau), \ldots, \chi_{n,n+1}(\tau)$ are the indicator functions defined by

$$\chi_{n,k}(\tau) := \mathbf{1}\left[L_{n,k} > \tau\right], \quad k = 1, \ldots, n+1.$$

The graph $\mathbb{G}(n;\tau)$ is connected if and only if $C_n(\tau) = 0$, and we have the representation

$$P(n;\tau) = \mathbf{P}\left[C_n(\tau) = 0\right]. \tag{17}$$

The basic idea of the proof is to leverage the representation (17) in order to provide lower and upper bounds on the probability of graph connectivity through moments of the counting variable $C_n(\tau)$: The method of first moment [12, Eqn. (3.10), p. 55] yields the inequality

$$1 - \mathbf{E}\left[C_n(\tau)\right] \leq P(n;\tau) \tag{18}$$

while the method of second moment [12, Remark 3.1, p. 55] gives the bound

$$P(n;\tau) \leq 1 - \frac{\mathbf{E}\left[C_n(\tau)\right]^2}{\mathbf{E}\left[C_n(\tau)^2\right]}. \tag{19}$$

With the help of (8) it is a simple matter to derive the closed-form expressions

$$\mathbf{E}\left[C_n(\tau)\right] = (n-1)(1-\tau)_+^n$$

and

$$\mathbf{E}\left[C_n(\tau)^2\right] = \mathbf{E}\left[C_n(\tau)\right] + (n-1)(n-2)(1-2\tau)_+^n.$$

Now, pick any range function $\tau : \mathbb{N}_0 \to \mathbb{R}_+$ of the form (2). We shall show below that

$$\lim_{n\to\infty} \mathbf{E}\left[C_n(\tau_n)\right] = 0 \quad \text{if} \quad \lim_{n\to\infty} \alpha_n = \infty \tag{20}$$

and

$$\lim_{n\to\infty} \frac{\mathbf{E}\left[C_n(\tau_n)^2\right]}{\mathbf{E}\left[C_n(\tau_n)\right]^2} = 1 \quad \text{if} \quad \lim_{n\to\infty} \alpha_n = -\infty. \tag{21}$$

These results readily ensure the validity of the one-law and zero-law upon letting n go to infinity in (18) and (19), respectively, where τ has been replaced by τ_n.

From Lemma 3.2, we readily see that

$$\lim_{n\to\infty} \mathbf{E}\left[C_n(\tau_n)\right] = \begin{cases} 0 & \text{if } \lim_{n\to\infty} \alpha_n = +\infty \\ & \\ \infty & \text{if } \lim_{n\to\infty} \alpha_n = -\infty. \end{cases} \tag{22}$$

Next, from the expressions given earlier, we conclude that

$$\frac{\mathbf{E}\left[C_n(\tau_n)^2\right]}{\mathbf{E}\left[C_n(\tau_n)\right]^2} = \mathbf{E}\left[C_n(\tau_n)\right]^{-1} + \frac{(n-2)}{(n-1)} \frac{(1-2\tau_n)_+^n}{(1-\tau_n)_+^{2n}}. \tag{23}$$

We have already shown that $\lim_{n\to\infty} \mathbf{E}\left[C_n(\tau_n)\right] = \infty$ whenever $\lim_{n\to\infty} \alpha_n = -\infty$. From Lemma 3.1 (first with $p = 2$ and then $p = 1$) under this last condition, we also get

$$\lim_{n\to\infty} \frac{(1-2\tau_n)_+^n}{n^{-2}e^{-2\alpha_n}} = \lim_{n\to\infty} \frac{(1-\tau_n)_+^n}{n^{-1}e^{-\alpha_n}} = 1.$$

It is now a simple matter to check from these facts that

$$\lim_{n\to\infty} \frac{(1-2\tau_n)_+^n}{(1-\tau_n)_+^{2n}} = \lim_{n\to\infty} \frac{(1-2\tau_n)_+^n}{n^{-2}e^{-2\alpha_n}} \cdot \left[\frac{n^{-1}e^{-\alpha_n}}{(1-\tau_n)_+^n}\right]^2 = 1$$

and (21) readily follows upon letting n go to infinity in (23). ∎

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